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BIOGRAPHY.

MR. W. J. C. MILLER.

BY B. F. FINKEL.

THE native place of Mr. Miller is in one of the most beautiful parts of the South coast of England. Of this district he has given a sketch in an article (in Nature-Notes for August, 1894) entitled a "Devonian Headland," which he describes as lying deep within the great West Bay of Dorset and Devon, and to which sea-birds have always flocked as to a chosen retreat. The upland chalk downs end in lofty cliffs that run sheer to the water's edge: and close by, both east and west, clear brooks, which spring from the underlying green-sand, have worn out charming little valleys that bear the Celtic name of combes. The headland itself bears a Norse name, derived from a village that lies in the eastern valley,—it was a little way off on the shores of the same great bay, that the Norsemen had their first historic conflict with the English—but the village itself might well bear a similar place-name with its western neighbor, and be called more appropriately, Chalcombe.

The district was a perfect paradise of birds, with which he became perfectly familiar as a boy, and on which, in later life, he loved to write articles describing their various habits. The mobbing, by a mingled flock of rooks and jackdaws, of a pair of ancient ravens that had built for ages in a neighboring cliff, till, at last, the powerful ravens, worn out by numbers, would find shelter from their tormentors in some wood, or cleft, or cave: the motions and chirpings of the stone-chats and the win-chats in the furze-bushes: the swift and dazzling flight of the king-fisher: the finding of the habitat of the dipper or water-ouzel——a song-bird that dives, and wades, and swims—watching its motions under water, and finding its nest year after year in the same stream: and the delight-

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ful turr-turr of the turtle-doves in the woods, reminding him that, for six months, he might say, with Virgil's Melibœus,

"Nec gemere aeria cessabit turtur ab ulmo":

these and many such sights and sounds he was ever delighted to recall and record.

In such pleasant regions as these, Mr. Miller was born, on August 31, 1832 ; here he roamed as a boy, always fond of books ; and amid these scenes he acquired that love of Nature, and especially of bird-life, that never afterwards deserted him. From the village-school he went to the Independent College of Taunton in Somerset, where he had for a time, the teaching of a distinguished Mathematician ; and from there he matriculated with mathematical honors at the University of London. Then came the great disappointment of his life. He was desirous to enter the great mathematical University of Cambridge ; but his parents belonged to the sect that had trampled down King, Church, and Aristocracy, one after the other ; that had formed an army that had never met, either in the British Islands or on the Continent, an enemy that could stand its onset ; and had sent across the Atlantic a band which, fleeing from persecution, had founded the third home of the great English race. Thus they could not endure that a son of theirs should submit to the tests then imposed in the University ; so he had to give it up. Years afterward, he learned from eminent mathematicians, that the best of all science was learned by one's own self, and never derived from any Professors at College or University. But he would then have gladly submitted to any test, if he had been allowed.

So he turned to study and instruction in mathematics ; and after teaching at various Institutions, became finally Professor at Huddersfield College in Yorkshire, where he remained many years, till he took the post that he now holds. There it was that he devised, and, after many trials, got a Publisher to undertake, the series of Volumes that he has edited ever since, and of which he is now engaged upon the sixty-fifth Volume. It was in 1861 that he conceived the idea of devising some plan whereby the contributions to the mathematical columns of ths *Educational Times*, which had been for some years under his Editorship, might be presented, apart from other matter, in a more convenient form than could be furnished by the pages of the Journal ; and, after ascertaining the views of his contributors, and obtaining promises of support, the mathematical solutions that appeared in each number were, from Midsummer, 1863, printed off, in the narrow columns then in use, from the journalistic types ; and at the end of a year the collection was, in July, 1864, issued as the first of the series. By and by the narrow columns were altered to wider columns ; and then the contributors were not content to wait a year for their articles : thus, ultimately, the issues took place at half-yearly intervals.

The series that took its rise from such small beginnings has gone on continuously from that time to this ; and is going on still. After 25 years it was necessary to issue a second edition of the first volume ; and this was brought out with improvements, uniform with the other Volumes, in the wider columns, in

1886. In these Volumes there have appeared, from time to time, articles in almost all branches of Mathematics, and the leading Mathematicians of all countries have continuously helped the work forward. One valued contributor, among early ones, was Dr. Hirst, F. R. S., who developed, in various articles, those elegant branches of Geometry in which all took a deep interest; and who, at last, collected and published his contributions in a separate Volume. Other important contributors to the early Volumes were Professor Cayley, to whom many articles were due; and the too-early lost Professor Clifford, who, being a fellow Devonian with the Editor, began to write when he was flying kites, and continued to furnish articles that increased in number and value through many volumes, accompanied by letters to the Editor that contained comments and developed views that were often more interesting than the articles themselves. The comparatively new theory of Local Probability was largely developed in the early Volumes by such writers as Woolhouse, Clarke, Crofton, Stephen Watson, our countryman, the late Professor E. B. Seitz who was a great master of difficult Probability Problems, and others. These contributors have all passed into the silent land. From a contributor who, it is to be feared, is getting near it, Professor Sylvester, articles followed in such quick succession that, from the very earliest times, there were but three or four numbers of the Journal, and those through the merest inadvertence, that did not contain, till the very last, at least one of his articles.

In 1876, Mr. Miller obtained the highly responsible post of Executive Officer (Registrar and Secretary) of the General Medical Council, an office in which he has remained ever since, continuing to edit, at his leisure, the mathematical periodical that has now attained to its 65th Volume. Among Editors of Mathematics this is deemed to establish for Mr. Miller what is termed "a Record": seeing that no other Mathematical Editor has ever, it is believed, gone on so long, with such laborious work as this. Always interested in Literature, no less than in Science, he edited for his Students at Huddersfield College a Magazine in which there came forward young contributors who afterward attained to eminence, whereof one has recently written an able book on the geography and resources of Africa.

During this time, he has been living in the finest of all the suburbs of London, in that Richmond whose name has been transferred to many other places, notably to that city which figured so largely in the Civil War. Under the title of "a Bird-loved Suburb of London" he has written an article to set forth its Bird-life, and its many beauties.

Here he founded in 1887, a Literary Society of which he is still President, and before which, on March 20th, one of his Mathematical contributors, Mr. George Heppel, M. A., lectured on the "Origins of European Poetry." In the course of his introductory remarks that evening, the chairman, Mr. Miller, said, "We are this evening entering upon a new departure. Hitherto, the lecturers have been members of our own society, but, in bringing in now for the first time a lecturer from outside, we are adopting a course that might hereafter

be worked out with advantage by our able and energetic secretary. Mr. Heppel is a mathematician, and such men have long been found peculiarly sensitive to the influence of the sister-arts of music and poetry. The very greatest of all living mathematicians [Professor Sylvester] called the attention of the Royal Society, twenty-five years ago, to the coincidence or parallelism, which observation has long made familiar, between the mathematical and the musical ethos : music being the mathematics of the sense, mathematics the music of reason ; the soul of each the same. Music the dream, mathematics the working life ; each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Beethoven-Gauss."

Other doings of Mr. Miller's during his life in Richmond, and his official duties at the General Medical Council, are set forth in the following article from the *Richmond and Twickenham Times* for August 17, 1889 :—

"Those who attended the meetings of the Richmond Athenæum, and the far larger number who read the reports of the proceedings of that body, are familiar with the pleasant, gracefully worded, and often erudite little speeches of Mr. W. J. C. Miller, a member of the council who has always been, in a double sense, a right-hand man to the chairman, sitting upon his right on the platform, and always ready, however abtruse the subject, to save a debate from flagging by filling up the regulation ten minutes with remarks which are always appropriate, often profound, invariably couched in the happiest words, and abounding in quotations from the poets, displaying a memory which is the admiration of all. Comparatively few, however, in Richmond know of the laborious and difficult duties in the world of mathematics to which Mr. Miller has devoted himself for more than thirty years, as editor of the *Educational Times*, or of the position which he has filled for thirteen years as the sole executive officer (registrar and secretary by name) in the management of the business of the General Medical Council.

With regard to Mr. Miller's editorial duties, many eminent mathematicians have given ungrudging testimony to their value. Thus Professor Sylvester—the first of living mathematicians—speaks of him as "an excellent mathematician, extensively and critically versed in all parts of the science, a good writer and lecturer on various subjects of natural science and other parts of human knowledge lying outside his own more special pursuits, and a most able and painstaking editor. . . . His scientific attainments are of a high order; he is deeply skilled in nearly all the departments of the highest mathematics, and is a novice in none. His labour as mathematical editor of the *Educational Times*, in which his own original papers are fit company for those of our foremost analysts, is proof of that. It would be a mistake to suppose him a mere schoolmaster or a mere mathematician. He is a sound classical scholar, and an erudite man of letters." The late lamented Professor Clifford considered that the mathematical portion of the *Educational Times* "has done more to suggest and encourage original research than any other European periodical." Equally gratifying words are used by Sir Robert Ball, Professor Tait, Dr. Hirst, Rev. Dr. Booth, Professor Crofton, Colonel Clarke, Dr. S. T. Hall, Professor Townsend, Professor Young, Dr. Todhunter, Rev, George Salmon, Professor Cayley, Professor Everett, and others whose attainments have raised them to the highest eminence. It has often been said that by Mr. Miller's mathematical work, the culture and study of the science have been more advanced than by any two or three agencies put together, in any or all parts of the world. When he commenced this important work he had but what was then an utterly obscure and almost unknown journal to use as his means of intercommunication and publication. Now he has nearly five hundred vigorous contributors, from all parts of the globe. Many are educated Hindoos (professors and others); many are Americans or Australians; still more are Germans, Frenchmen, Russians, or Italians; some are Spaniards; and some write from the South American Republics.

The multifarious work of the General Medical Council has more than quadrupled since Mr. Miller took it in charge thirteen years ago. Established to carry out the voluminous Medical Acts (which cover fifty-nine pages of the *Medical Register*), the Council had to take charge, in 1878, of all the dentists in the empire, and since then of various other matters, including, quite recently, the registration of sanitary officers. Many testimonies to the appreciation of Mr. Miller's services have been given by the Council, and by the medical newspapers. Thus the *Medical Press*, in a recent article on the General Medical Council, says that—"Every session marks a distinct improvement in the business aptitude of the Council, and in the amount of work accomplished, results which may fairly be attributed in no small degree to a more vigorous presidential control, and to the efficiency of the business arrangements, which depend so much for their success on the services of a competent and attentive registrar." The *British Medical Journal*, in reviewing the *Minutes of the General Medical Council*, says—"The Volume has been edited by Mr. W. J. C. Miller, B. A., the Registrar of the General Council, with the care which he has accustomed us to expect from him." The *Report of the Statistical Committee of the General Medical*

Council is another work of which Mr. Miller has charge, and in noticing this the *Medical Press* says—"We assume that its compilation is chiefly due to the energy and noted mathematical skill of Mr. Miller, the Registrar of the Council, and if we are correct in this assumption we can only remark that both the profession and the Medical Council owe that gentleman much thanks for work which, though no doubt a labour of love, must involve great devotion of time and mental capacity." Another work of the utmost importance to the public, and for the annual publication of which Mr. Miller is responsible, is the *Medical Register*, which has now grown to a volume of 1,198 pages. In addition to this there is the *Dentist's Register* (232 pages), besides the *Medical Students' Register*, the latter alone requiring 100 pages. It would be difficult to speak too highly of the care exhibited in the compilation of these important works. Referring to the issues for the present year, the *Medical Press* says "They display all the progressive improvement which has been manifested since Mr. Miller took them in hand."

Being an ardent lover of science and literature, Mr. Miller has all his life striven to aid others in sharing their delights, by lectures, writings, and teaching. And all this work, editorial and other, has been not only unremunerative, but carried on with no little outlay. But the world's best workers have always been the most unselfish. Mr. Miller has at least the gratification of knowing that his favourite pursuits have been greatly advanced by his efforts, and that he has earned the gratitude of many who have reaped the advantages of his self-denying work."

Mr. Miller was one of the earliest members of the London Mathematical Society; but as he found that, with his official duties, and his Editorial work, he could not spare time to attend the Meetings, he was reluctantly compelled to resign his membership. Since that time, he has had to devote the whole of his small leisure to the duties of his Editorship, which goes on increasing every month, with new contributors from foreign countries, especially India, where an enlarged interest is rapidly growing in all the articles that are published in his Journal. Mr. Miller is a great admirer of America and American ways of managing; he entertains a high opinion of our magazine, and says it is one of the best that comes to him. He has a large circle of friends and admirers in America, most of whom are contributors to the Mathematical Department in the *Educational Times.*

THE EXPONENTAL DEVELOPMENT FOR REAL EXPONENTS.

By WILLIAM BENJAMIN SMITH. Ph. D., (Görtingen) Professor of Mathematics, Tulane University, New Orleans, Louisiana.

The Exponental Series is of such fundamental and far-reaching importance, it is so indispensable to all higher Analysis, that it seems strange so few if any deductions of it accessible to the English reader should be carefully conducted; not even that given by Chrystal in his superb *Treatise on Algebra* can lay claim to rigor. It may be worth while then, under no pretense of novelty, to attempt to supply this lack in some measure.

I. We consider the expansion given by the Binomial Theorem:

$$\left(1+\frac{x}{n}\right)^n=1+n.\frac{x}{n}+\frac{n(n-1)}{\underline{|2}}.\frac{x^2}{n^2}+\frac{n(n-1)(n-2)}{\underline{|3}}.\frac{x^3}{n^3}+\ \ldots\ldots$$

$$=1+x+\left(1-\frac{1}{n}\right).\frac{x^2}{\underline{|2}}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right).\frac{x^3}{\underline{|3}}+\ldots\ldots,$$

where x is finite and positive, while n is *positive* and *integral*, and we inquire whether this series tends toward a definite form and value as n increases without limit.

We denote the differences $1-\frac{1}{n}$, $1-\frac{2}{n}$, ... $1-\frac{k}{n}$ ...... by d_1, d_2, ...d_k, ... and the products of these, d_1, $d_1 d_2$, $d_1 d_2 d_3$, ...$d_1 d_2$ ......d_k, ...by p_1, p_2, p_3, ...p_k, ...

Then plainly $1>d_1>d_2>d_3>\ldots\ldots>d_k>\ldots\ldots$

and also $1>p_1>p_2>p_3>\ldots\ldots>p_k>\ldots\ldots$

In the expansion there are $n+1$ terms, which we may write t_0, t_1, t_2, ...t_r, ...t_n. We consider the sum $t_0+t_1+\ldots\ldots+t_r$ and denote it by S_r; then the sum of remaining $n-r$ terms we denote by V_r, so that $\left(1+\frac{x}{n}\right)^n=S_r+V_r$ where

$$S_r=1+x+p_1\frac{x^2}{\underline{|2}}+p_2\frac{x^3}{\underline{|3}}+\ldots\ldots+p_k.\frac{x^{k+1}}{\underline{|k+1}}+\ldots\ldots+p_{r-1}.\frac{x^r}{\underline{|r}},$$

$$V_r=p_r.\frac{x^{r+1}}{\underline{|r+1}}+p_{r+1}.\frac{x^{r+2}}{\underline{|r+2}}+\ldots\ldots+p_{n-1}.\frac{x^n}{\underline{|n}}.$$

Since n is to be taken *great at will*, r may also be taken *great at will* and yet *always less than* n. We now ask, what becomes of S_r as r increases without limit while always $r<n$? Since all the p's are <1, it is plain that

$$S_r<\left\{1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ \ldots\ldots+\frac{x^r}{\underline{|r}}\right\}.$$

However we can make *each* of the p's$>1-\sigma$, where σ is *small at will*. It is enough to prove this for the *least* of the p's, p_r. We have

$$p_r=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\ldots\left(1-\frac{r-1}{n}\right)>\left(1-\frac{r-1}{n}\right)^{r-1}.$$

Now $\left(1-\frac{r-1}{n}\right)^{r-1}=1-\sigma$ if $1-\frac{r-1}{n}=(1-\sigma)^{\frac{1}{r-1}}$, or if $\frac{r-1}{n}=1-(1-\sigma)^{\frac{1}{r-1}}$, or if

$$n \overline{\gtrless} \frac{r-1}{1-(1-\sigma)^{\frac{1}{r-1}}}.$$

Now for any finite value of r *however great*, and for any finite value of σ *however small*, this fraction on the right will always be finite and perfectly difinite, though perhaps very great; hence it will always be possible to choose n equal or even greater (in case the fraction be not integral in value); hence it will always be possible to make $p_r > 1-\sigma$, no matter how great r or how small σ, by merely choosing n great enough, and this we can always do, since n is quite at our will.

$$\text{Hence } S_r > (1-\sigma)\{(1+x+\frac{x^2}{\underline{|2}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

$$\text{Hence } \{1+x+\frac{x^2}{\underline{|2}}+\ldots\ldots+\frac{x^r}{\underline{|r}}\} > S_r > (1-\sigma)\ \{1+x+\frac{x^2}{\underline{|r}}+ \ldots\ldots + \frac{x^r}{\underline{|r}}\}.$$

Hence S_r differs from $\{\ldots.\}$ by less than $\sigma\{\ldots.\}$. Now this brace, $\{\ldots.\}$ is of course finite for all finite values of r, and it also remains finite (for x finite) even for r increasing without limit. For the ratio of two consecutive terms is $\frac{x}{k}$ and this ratio is not only finitely <1 for $k>x$ but it becomes ever smaller and smaller, sinking below every assignable degree of parvitude as k increases without limit, x of course being finite and fixed, however great. Hence $\sigma\{\ldots\ldots\}$ is small at will, since the product of a magnitude small at will multiplied by a finite number is itself small at will. Hence the two magnitudes $\{\ \ldots..\}$ and $(1-\sigma)\{\ldots..\ \}$ close down upon each other as r increases without limit, hence they close down upon S_r always between them, so that we have

$$\text{Lim.} S_r = 1+x+\frac{x^r}{\underline{|r}}+\ \ldots\ldots + \frac{x^r}{\underline{|r}}$$

for r and n increasing without limit, $n>r$. It remains to examine V_r. We may

$$\text{write } V_r = t_{r+1}(1+d_{r+1}\ \frac{x}{r+2}+d_{r+1}d_{r+2}\ \frac{x^2}{(r+2)(r+3)}+\ \ldots.\ ).$$

Now $t_{r+1} < \frac{x^{r+1}}{\underline{|r+1}}$ and this we have just seen is small at will for r great at will, or $t_{r+1} < \sigma$. Also the parenthesis $(\ \ldots.\ ) < [1+\frac{x}{r+2}+\frac{x^2}{(r+2)^2}+\ldots.]$,

and this bracket $[\ldots\ldots]$ is finite for all finite values of n and r, and it has 1 for its limit as n and r increase without limit. Hence $V_r < \sigma$, or Lim. $V_r = 0$, hence

$$\text{Lim.}\,(1+\frac{x}{n})^n = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots \textit{ in infinitum.}$$

II. Thus far n has been integral at every stage of value. What if it be *fractional* or *irrational?* The preceding proof does not then apply but we shall always have n lying between two consecutive integers, or $w<n<w+1$.

Now $(1+\frac{x}{n})^n < (1+\frac{x}{w})^{w+1}$, a greater number raised to a higher power; or

$(1+\frac{x}{n})^n < (1+\frac{x}{w})^w\,(1+\frac{x}{w})$. Likewise $(1+\frac{x}{n})^n > (1+\frac{x}{w+1})^w$, a smaller number to a lower power. Or $(1+\frac{x}{n})^n > (1+\frac{x}{w+1})^{w+1} / (1+\frac{x}{w+1})$.

Now for w and $w+1$ increasing without limit we have just proved that $(1+\frac{x}{w})^w$ and $(1+\frac{x}{w+1})^{w+1}$ both close down upon one and the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$$

Also the multiplier $1+\frac{x}{w}$ and the divisor $1+\frac{x}{w+1}$ both close down upon the same limit 1; hence the product $(1+\frac{x}{w})^w.(1+\frac{x}{w})$ and the quotient $(1+\frac{x}{w+1})^{w+1} / (1+\frac{x}{w+1})$ both close down upon the same Limit,

$$1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots;$$

hence $(1+\frac{x}{n})^n$ lying always between this product and this quotient itself closes down upon the same limit; hence $\text{Lim.}(1+\frac{x}{n})^n = 1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots\ldots$ for finite positive x and for positive n increasing no matter how without limit.

III. For x *negative* we must consider $(1-\frac{x}{n})^n$. This we write $=E-O$, a sum of even powers, less a sum of odd powers, of x. Each of these we break up into two parts, S_r and V_r, and reiterate the foregoing argument with insignificant and self-evident modifications. There results

$$\text{Lim.}(1-\frac{x}{n})^n = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \ldots\ldots$$

for finite positive x and n increasing no matter how without limit. Or

$$\text{Lim.}(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$$

for any finite real x positive or negative, n increasing any way without limit.

IV. For n *negative* we have

$$\left(1-\frac{x}{n}\right)^{-n} = \left(\frac{n-x}{n}\right)^{-n} = \left(\frac{n}{n-x}\right)^n = \left(1+\frac{x}{n-x}\right)^n =$$

$$\left(1+\frac{x}{n-x}\right)^{n-x} \cdot \left(1+\frac{x}{n-x}\right)^x = \left(1+\frac{x}{n-x}\right)^{n-x} \Big/ \left(1-\frac{x}{n}\right)^x.$$

Now as n increases without limit, so also does $n-x$, for x finite no matter how great; hence $(1+\frac{x}{n-x})^{n-x}$ approaches as its limit the series $1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ldots$ and $(1-\frac{x}{n})^x$ approaches 1 as its limit manifestly; hence

$$\text{Lim.}(1-\frac{x}{n})^{-n} = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$$

Hence $\text{Lim.}(1+\frac{x}{n})^n = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$ *for all finite real values of x, for real n increasing without limit no matter how, positively or negatively.*

For $x=1$ we obtain $\text{Lim.}(1+\frac{1}{n})^n = 1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|5}}+\ldots.$

The number defined by this series and denoted by e, is one of the three irrationals (π, i, e) all-important to analysis. That e is irrational may be easily seen thus: Consider the first $(p+1)$ terms $1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\ldots\ldots\frac{1}{\underline{|p}}$; the sum Σp is $\frac{w}{\underline{|p}}$ where w is some integer no matter what; the remainder

$$\frac{1}{\underline{|p+1}} + \frac{1}{\underline{|p+2}} + \ldots\ldots \quad \text{is} \frac{1}{\underline{|p}}\left\{\frac{1}{p+1} + \frac{1}{(p+1)(p+2)} + \ldots\ldots\right\}$$

$$< \frac{1}{\lfloor p}\left(\frac{1}{p+1} = \frac{1}{(p+1)^2} + \ldots\ldots\right) < \frac{1}{\lfloor p}. \quad \text{Hence } \frac{w}{\lfloor p} < e < \frac{w+1}{\lfloor p}.$$

Now as p increases without limit these two fractions close down upon each other and upon e always between them. It is plain that there is no fixed fraction as N/D, always between $\frac{w}{\lfloor p}$ and $\frac{w+1}{\lfloor p}$; for however great D might be, we could choose p so large, that $\lfloor p$ would include all the factors of D; hence $\frac{N}{D} = \frac{w}{\lfloor p}$, whereas $e > \frac{w}{\lfloor p}$. In fact as the two fractions $\frac{w}{\lfloor p}$ and $\frac{w+1}{\lfloor p}$ close down on each other and on their common limit e they *pass over* (either the one or the other) every assignable fraction lying between them.

The importance of e lies in the fact that the series $1+x+\frac{x^2}{\lfloor 2}+\ldots\ldots$ is expressible through it. We have

$$(1+\frac{x}{n})^n = \{(1+\frac{x}{n})^{n/x}\}^x = \{(1+\frac{1}{n/x})^{n/x}\}^x. \quad \text{Hence}$$

$$\text{Lim.}(1+\frac{x}{n})^n = \text{Lim.}\{(1+\frac{1}{n/x})^{n/x}\}^x = \{\text{Lim.}(1+\frac{1}{n/x})^{n/x}\}^x,$$

where we indeed assume that the Limit of the Power equals the Power of the Limit. But this is plainly correct, at least in the present case; for

$$(1+\frac{1}{n/x})^{n/x} = e+\sigma.$$

Hence $\text{Lim.}\{(1+\frac{1}{n/x})^{n/x}\}^x = \text{Lim.}(e+\sigma)^x = \text{Lim.}(e^x+\sigma x e^{x-1}+\ldots\ldots) = e^x$ for all finite real values of x. Hence

$$\text{Lim.}(1+\frac{x}{n})^n = e^x = 1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots \textit{ in infinitum.}$$

Herewith then the exponental development is established for all finite real values of the exponent.

Tulane University, May, 1896.

NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJ. F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from April Number.]

VII. Let ABC be a $\triangle$ right angled at C. Produce BC making $BD=BA$. Join DA. From E, the middle point of CD, draw a perpendicular meeting DA, as at F. Join FB. $\triangle ADC$ is similar to $\triangle BFE$.

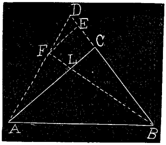


Fig. 7.

$\therefore AC : BE :: DC : FE$.

$\therefore AC : BC+(AB-BC)\div 2 :: AB-BC : AC\div 2$.

$\therefore \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

NOTE.—This proof is credited to Hoffman.

VIII. Triangles BDF, BFE, and FDE, are similar.

Letting $BD=BA=c$, $AC=b$, $BC=a$, $BE=\frac{a+c}{2}$, $DE=\frac{c-a}{2}$, $FE=\frac{b}{2}$, $DF=x$, $BF=y$, we obtain the following:

(1). $\frac{c-a}{2} : x :: x : c$. $\therefore x^2=\frac{c(c-a)}{2}$ ....1.

(2). $\frac{c-a}{2} : x :: \frac{b}{2} : y$. $\therefore bx=(c-a)y$ ....2.

(3). $x : c :: \frac{b}{2} : y$. $\therefore xy=\frac{bc}{2}$ ....3.

(4). $\frac{c-a}{2} : \frac{b}{2} :: \frac{b}{2} : \frac{c+a}{2}$. $\therefore c^2-a^2=b^2$ ....4.

(5). $\frac{c-a}{2} : \frac{b}{2} :: x : y$. $\therefore bx=(c-a)y$ ....2.

(6). $\frac{b}{2} : \frac{c+a}{2} :: x : y$. $\therefore by=(c+a)x$ ....5.

(7). $\frac{c+a}{2} : y :: y : c$. $\therefore y^2=\frac{c(c+a)}{2}$ ....6.

(8). $\frac{c+a}{2} : y :: \frac{b}{2} : x$. $\therefore by=(c+a)x$ ....5.

(9). $y : c :: \frac{b}{2} : x$. $\therefore xy=\frac{bc}{2}$ ....3.

From 4 we get $c^2=a^2+b^2$.

The set 2 and 5 gives the same result. But equation 2 may come from proportion (2) or (5), and 5 from (6) or (8), thus making four proofs for this set.

The following sets of three equations furnish fourteen proofs, since each set can come from two or more sets of three proportions: 1, 2, 6 ; 1, 3, 5 ; 1, 3, 6 ; 1, 5, 6 ; 2, 3, 6. Total number of proofs for this method is 19.

IX. Comparing the triangles BDF, BFE, ADC, BLC, and ALF, Fig. 7, we may put the result in the following condensed form:

$$DF=x : EF=\tfrac{1}{2}b : DC=c-a : LC=z : FL=y-v$$

$$:: FB=y : EB=\tfrac{1}{2}(c+a) : AC=b : CB=a : AF=x$$

$$:: BD=c : FB=y : AD=2x\ LB=v : AL=b-z.$$

From this we easily may derive thirty different simple proportions, which give twenty-seven different equations. Some idea of the number of proofs that may be obtained from different sets of these equations, can be formed from the fact that there are 17550 sets of four equations, to say nothing of sets of three and of five. Of course, many of the sets must be rejected for reasons stated fully in V. We leave details to the reader.

X. Suppose the theorem true. Then $\overline{AB}^2=\overline{AC}^2+\overline{BC}^2$, $\overline{BC}^2=\overline{CD}^2+\overline{BD}^2$, and $\overline{AC}^2=\overline{AD}^2=\overline{CD}^2$.

$\therefore\ \overline{AB}^2=\overline{AD}^2+2\overline{CD}^2+\overline{BD}^2$.

But $\overline{CD}^2=AD\cdot BD$.

$\therefore\ \overline{AB}^2=\overline{AD}^2+2AD\cdot BD+\overline{BD}^2$.

$\therefore\ AB=AD+BD$, which is true.

$\therefore$ The supposition is true.

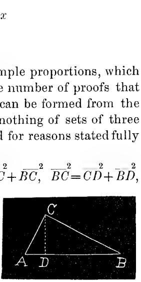


Fig. 1.

NOTE.—This method is credited to Hoffman.

XI. In Fig. 1, $\overline{AB}^2<, =,$ or $> \overline{AC}^2+\overline{BC}^2$. Suppose it less. Then, since $\overline{AB}^2=(AD+DB)^2=(\overline{CD}^2\div DB+DB)^2$, and $\overline{AC}^2=(\overline{CD\cdot BC}\div DB)^2$,

$$(\overline{CD}^2\div DB+DB)^2<(\overline{CD\cdot BC}\div DB)^2+\overline{BC}^2.$$

$\therefore\ (\overline{CD}^2+\overline{DB}^2)^2<\overline{BC}^2(\overline{CD}^2+\overline{DB}^2)$.

$\therefore\ \overline{BC}^2>\overline{CD}^2+\overline{DB}^2$, which is absurd. For were the supposition true, we should have $\overline{BC}^2<\overline{CD}^2+\overline{DB}^2$, as can easily be shown.

Similarly the supposition that $\overline{AB}^2>\overline{AC}^2+\overline{BC}^2$ can be proven false.

$\therefore\ \overline{AB}^2=\overline{AC}^2+\overline{BC}^2$.

XII. $BC=a : EF=x : DF=y : DE=z$

$:: AC=b : AF=v : EF=x : AE=w$

$:: AB=c : AE=w : ED=z : AD=v+y.$

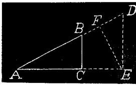


Fig. 8.

The above condensed form is self-explanatory, as are also the two following.

We leave the selection of simple proportions, the derivation and solution of consequent equations, as an exercise for the interested reader.

XIII. $BC=a : DE=x : DL=y,$

$:: AB=b : AE=z : LF=FE=v,$

$:: AB=c : AD=v+y : DF=x-v.$

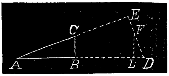


Fig. 9.

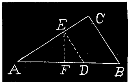


Fig. 10.

XIV. $BC=a : ED=EC=x : FD=y : EF=z,$

$:: AC=b : AE=b-x : EF=z : AF=v,$

$:: AB=c : AD=v+y : ED=x : AE=b-x.$

[To be Continued.]

INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from May Number.]

Each of the regular groups of degree six contains only one subgroup of the type $(abc.def)$. Since no substitution of the form $abcde$ can transform this into itself the group of order 30 is impossible.

If a group of order 60 exists it must contain six subgroups of the type $(abcde)_{10}$. We may assume that it contains $(abcde)_{10}\equiv G_f$. G_b must then contain $ac.de$ and a substitution of the type $abcde$ which contains the letters a, c, d, e, f. We may assume that this substitution is $acd_1e_1f_1$. It is then necessary that $ac.de.acd_1e_1f_1=af_1e_1d_1c.ac.de$. Hence

$$acd_1e_1f_1=acdfe \text{ or } acefd.$$

Since $acefd.adbec = bef$ every group of order 60 must contain

$$(abcde)_{10} \text{ and } acdfe.$$

These substitutions generate a group whose order $\gtreqless 60$, hence only one group of order 60 is possible.

We shall prove that these substitutions generate a group of order 60 by employing a very elementary but somewhat lengthy method. Representing the substitutions of $(abcde)_{10} \equiv 1$, $abcde$, $acebd$, $adbec$, $aedcb$, $ab.ce$, $ac.de$, $ad.bc$, $ae.bd$, $be.cd$ respectively by $1 = s_1, s_2, s_3, s_4, s_5, c_6, s_7, s_8, s_9, s_{10}$ and $acdfe$ by t, we form the rectangle

$$\begin{array}{llll}
1 & s_2 & s_3 & \ldots\ldots\ldots\ldots s_{10} \\
t & s_2 t & s_3 t & \ldots\ldots\ldots\ldots s_{10} t \\
t^2 & s_2 t^2 & s_3 t^2 \ldots & \ldots\ldots\ldots s_{10} t^2 \\
t^3 & s_2 t^3 & s_3 t^3 & \ldots\ldots\ldots\ldots s_{10} t^3 \\
t^4 & s_2 t^4 & s_3 t^4 & \ldots\ldots\ldots\ldots s_{10} t^4 \\
t_1 & s_2 t_1 & s_3 t_1 & \ldots\ldots\ldots\ldots s_{10} t_1
\end{array}$$

Where t_1 is any substitution generated by $(abcde)_{10}$ and $acdfe$ which is not found in the preceding five rows. These substitutions are all different. They form a group if $t_1{}^2$ is contained in the first five rows and

$$t^{\alpha} s_{\beta} = s_{\gamma} t^{\delta} \text{ or } s_{\gamma} t_1, \quad t_1 s_{\beta} = s_{\gamma} t^{\delta} \text{ or } s_{\gamma} t_1$$

$$(\beta, \gamma = 1, 2, \ldots\ldots 10), \ (\alpha, \delta = 1, 2, \ldots\ldots 4).$$

Instead of allowing β to have 10 values it is clearly sufficient to assign to it only the two values of 2 and 6 since $abcde$ and $ab.ce$ generate $(abcde)_{10}$. The following shows that the necessary conditions are fulfilled:

$$\begin{array}{ll}
t s_2 = adf.bce = s_{10} t^2 & t s_6 = aeb.cdf = s_3 t^3 \\
t^2 s_2 = aed.bcf = t_1{}^* & t^2 s_6 = adcfb = s_9 t_1 \\
t^3 s_2 = afdbc = s_9 t^4 & t^3 s_6 = afedb = s_4 t \\
t^4 s_2 = bc.ef = s_5 t & t^4 s_6 = acb.def = s_8 t^4 \\
t_1{}^2 = ade.bfc = s_5 t^3 & t_1 s_2 = bd.cf = s_4 t^3 \\
 & t_1 s_3 = acf.bed = s_9 t^2
\end{array}$$

There is therefore one group of order 60, viz:

(1) $$(abcde)_{10}(acdfe) = (abcdef)_{60}.$$

If there is a primitive group of order 120 it may be assumed that it con-

*In the above rectangle f is followed by the same letter as in the corresponding t or t_1. Since it is not followed by b in t, $aed.bcf$ cannot be contained in the first five lines and may therefore be used for t. All these relations may be readily found if this property is observed.

tains $(abcde)_{20}$ and therefore $(abcdef)_{60}$. Since half of its substitutions must be negative it must contain $(abcdef)_{60}$ as a self-conjugate subgroup.

The order of a group which satisfies these conditions cannot be less than 120. From this we see that there cannot be more than one group of this order. That there is one follows from the facts that *acbe* belongs to $(abcde)_{20}$ and transforms *acdfe* into *acbdf*—$(s_9 t_1)^2$—some substitution of $(abcdef)_{60}$.

The other primitive groups of degree six must contain subgroups of degree five which contain substitutions of one of the two types

ab *abc*

They must therefore be the alternating and the symmetric group. The following is therefore a complete list of these groups:

Order	Group
60	$(abcdef)_{60}$
120	$(abcdef)_{120}$
360	$(abcdef)$pos
720	$(abcdef)$*all*

REMARKS.

We have now finished the explanations of the elementary methods of group construction. By means of these we have been able to find, with a reasonable amount of labor, all the groups whose degree does not exceed six. It scarcely needs to be stated that this labor could have been considerably reduced by employing more advanced methods. In fact, we did not endeavor so much to find these groups by the least labor as to find them in such a way as to illustrate some of the most important elementary methods of group construction.

We are indebted to our honored teacher, Professor F. N. Cole not only for many of these methods but also for the fundamental ideas.

Most of the theorems that we have developed are found in Part I of Netto's Theory of Substitutions (American Edition). In some instances it seemed desirable to change the method of proof either because we had not yet developed the principles upon which Netto's proof is based, or because we desired to call attention to some special property. In a few instances our purposes required us to pursue the demonstration farther than is done by this author.

We did not enter into a special study of methods of operating with substitutions. Some of the more important ones have been incidentally explained. For further explanations we would refer to Senet's Algèbre Supérieure, Part IV, (this part is found in the second volume of this work), and to Part I of Netto.

In these works is also found considerable on the analysis of a substitution. The first 15 pages of the first volume of Gordan's Invariantentheorie contain considerable on this point. For the more advanced methods of operation we have

to refer to the classical work on this subject, Jordan's Traité des Substitutions, and to the periodicals.

Before entering upon the development of more advanced methods of group construction we shall study some of the relations which exist between substitution groups and functions containing a finite number of letters. These relations will not only show how substitution groups may be utilized but they may also serve as a means of arriving at important properties of substitution groups.

Leipzig, Germany, September 20, 1895.

SIMULTANEOUS QUADRATIC EQUATIONS.

By I. H. BRYANT, M. A., Instructor of Mathematics, Waco High School, Waco, Texas.

[Continued from May Number.]

The discussion in this article is restricted to two unknown quantities. Cases 1, 2, 4, and 5 apply to two variables just as they are stated in the previous article. In Case 3, the restriction that each factor must occur twice is unnecessary when only two variables occur. It is sufficient for one factor to occur in each equation. This reduces Case 3 to Case 2. For two variables, Cases 6, 7, and 8 become one and the same, as no restrictions are necessary.

The following Cases are applicable to two variables only. Express the equations thus for Cases 9 and 10:

$$ax^2+by^2+cxy+dx+ey+f=0. \qquad 1.$$

$$a'x^2+b'y^2+c'xy+d'x+e'y+f'=0. \qquad 2.$$

CASE 9. When $a:a'::c:c'::d:d'$. If this is true, it is obvious that the terms containing x can be eliminated. This holds true when the terms of any one, or any two, of the three ratios are zero.

CASE 10. When $a:a'::b:b'::d^2:d'^2::e^2:e'^2$, and when $d:d'::e:e'$.

By alternation, $\frac{e}{d}=\frac{e'}{d'}$, $\frac{b}{a}=\frac{b'}{a'}$.

Let $\frac{e}{d}=r$. Then $\frac{e'}{d'}=r$, $\frac{b}{a}=r^2$, $\frac{b'}{a'}=r^2$.

$e=dr,\ e'=d'r,\ b=ar^2,\ b'=a'r^2.$

Divide equation 1 by ad, and equation 2 by $a'd'$.

$$\frac{x^2+r^2y^2}{d}+\frac{x+ry}{a}+\frac{cxy+f}{ad}=0. \qquad \frac{x^2+r^2y^2}{d'}+\frac{x+ry}{a'}+\frac{c'xy+f^2}{a'd'}=0.$$

Let $x=r(u+v)$, $y=u-v$, and substitute. Eliminate v^2 and solve the resulting equation. The signs of d and e, and of d' and e' must be alike in both, or unlike in both equations. They cannot be alike in one and unlike in the other. Symmetrical equations are a special form of this case.

CASE 11. When the two equations can be expressed as follows:

$$(mx+p)^2-(m'x+p')^2+r(ny+q+n'y+q')=0$$

$$(ny+q)^2-(n'y+q')^2+r'(mx+p+m'x+p')=0.$$

By factoring, one value, and only one, of x and y can be found. The "Harvard Catch" is a special form of this Case.

ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

59. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

A broker charges me $1\frac{1}{2}$ per cent. brokerage for buying some uncurrent bank bills at 20 per cent. discount. Of these bills, 4 of \$50. each become worthless, but the remainder I dispose of at par, and make by the operation \$364. What was the face amount? [Which answer is correct, \$3000, or $\$3048\frac{24}{37}$?]

I. Solution by J. C. CORBIN, Pine Bluff, Arkansas; H. C. WILKES. Skull Run, West Virginia; and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

$80\%+1\frac{1}{2}\%=81\frac{1}{2}\%$; $100\%-81\frac{1}{2}\%=18\frac{1}{2}\%$.

$\$364+(4\times\$50)=\$564$, amount he would have made if he had disposed of all at par.

$\$564\div18\frac{1}{2}\%=\$3048\frac{24}{37}$, the correct face amount.

The other result is obtained as follows:

$1\frac{1}{2}\%$ of $80\%=1\frac{1}{5}\%$.

$80\%+1\frac{1}{5}\%=81\frac{1}{5}\%$; $100\%-81\frac{1}{5}\%=18\frac{4}{5}\%$.

$\$564\div18\frac{4}{5}\%=\$3000$.

The latter is commission on money invested and not brokerage on bills bought.

II. **Solution by F. M. McGAW, Bordentown, New Jersey.**

Market Value+Brokerage equals whole cost, therefore gain % was $1.00-(.80+.015)=.185$.

The net gain in money was \$364 to which we add the \$200 lost, making a gross gain of \$564. Then $18.5\%=\$564$, whence $\$564\div.185=\$3058\frac{24}{37}$, face.

To determine which answer is correct, assume the answer and work backwards.

I. Assume $\$3048\frac{24}{37}$ as face, then

$\$3048\frac{24}{37}\times.815=\text{cost}=\$2484\frac{24}{37}$,

$\$3048\frac{24}{37}-\$200(\text{lost})=\$2848\frac{24}{37}$

$\$2848\frac{24}{37}-\$2484\frac{24}{37}=\$364$, net gain. Answer.

II. Assume \$3000 as face, then the same operations produce a gain of only \$355.

Also solved by *A. P. REED*, *H. C. WHITTAKER*, *P. S. BERG*, and *J. SCHEFFER*.

We received solutions of problem 58, too late for credit in last issue, from *J. SCHEFFER*, *E. R. ROBBINS*, and *P. S. BERG*.

PROBLEMS.

62. **Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.**

A dealer buys milk at $m=5$ cents per quart, and sells it at $n=6$ cents per quart. How much water has he put with the milk, if his rate of profit is $p=60$ per cent.?

63. **Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.**

I owe A \$100 due in 2 years, and \$200 due in 4 years; when will the payment of \$300 equitably discharge the debt, money being worth 6 per cent.?

64. **Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.**

If 27 men in 10 days of 7 hours each for \$375 dig a ditch 70 rods long, 25 feet wide, and 4 feet deep, how long a ditch 40 feet wide and 3 feet deep will 15 men dig in 16 days of 9 hours each for \$500?

[77 2-9 rods and 88 8-9 rods have been obtained. Which is correct ?]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

60. Proposed by ROBERT J. ALEY, A. M., Professor of Mathematics in Indiana University, Bloomington, Indiana.

Telegraph poles are a yards apart; for how many minutes must one count poles in order that the number of poles counted may be equal to the number of miles per hour that the train is running?

I. Solution by FREDERICK R. HONEY, A. B., New Haven, Connecticut.

The problem is independent of the number of poles counted, and of the number of miles per hour the train is running. We will call this number N.

$\therefore aN$=the number of yards the train runs while the poles are counted. Also, $1760N$=number of yards per hour the train runs. $\therefore aN/1760N$=the fraction of an hour during which the poles are counted.

$\therefore 60aN/1760N=3a/88$=number of *minutes* during which the poles are counted.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; and W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana.

Let x=the number of minutes, and let r=number of miles per hour the train is running. Also, $1760/a$=number of poles in a mile, and $rx/60$=number of miles the train runs in x minutes. Then, $rx/60\times1760/a=88rx/3a$=number of poles passed in x minutes, or while the train is running $rx/60$ miles.

$\therefore 88rx/3a=r$; whence $x=3a/88$.

The number of minutes depends upon the distance the poles are apart irrespective of the rate of the train.

Also solved by *O. W. ANTHONY, P. S. BERG, A. H. HOLMES, C. D. SCHMITT, H. C. WILKES, B. F. YANNEY*, and *G. B. M. ZERR.*

61. Proposed by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

Demonstrate the identity $2^{2n+1}\dfrac{d^n}{dx^n}\left(x^{n+\frac{1}{2}}\dfrac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\right)=e^{\sqrt{x}}$.

I. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

It may be proved inductively that $\dfrac{d^n}{dx^n}e^{\sqrt{x}}=\dfrac{1}{4x}\,\dfrac{d^{n-2}}{dx^{n-2}}e^{\sqrt{x}}-(n-\tfrac{3}{2})\dfrac{d^{n-1}}{dx^{n-1}}e^{\sqrt{x}}$.

Change n to $n+3$; then

$$\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}=\frac{1}{4x}\,\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}-(n+\tfrac{3}{2})\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}.$$

Clearing of fractions and transposing,

$$\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}=4(n+\tfrac{3}{2})\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}+4x\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}.$$

Multiply through by $x^{n+\frac{1}{2}}$, and we have

$$x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}=4[(n+\tfrac{3}{2})x^{n+\frac{1}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}+x^{n+\frac{3}{2}}\frac{d^{n+3}}{dx^{n+3}}e^{\sqrt{x}}]$$

$$=4\frac{d}{dx}[x^{n+\frac{3}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}].$$

Multiply through by 2^{2n+1} and differentiate n times, and we have

$$2^{2n+1}\frac{d^n}{dx^n}(x^{n+\frac{1}{2}}\frac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}})=2^{2n+3}\frac{d^{n+1}}{dx^{n+1}}(x^{n+\frac{3}{2}}\frac{d^{n+2}}{dx^{n+2}}e^{\sqrt{x}}).$$

Hence, if the form given is true for n, it will be true for $n+1$. It may be easily verified that it is true for $n=2$. Therefore it is *generally* true.

II. **Solution by HENRY HEATON, M. S., Atlantic, Iowa, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.**

$$\frac{d}{dx}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}}{2\sqrt{x}},\ \frac{d^2}{dx^2}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(\sqrt{x}-1)}{4x^{\frac{3}{2}}},\ \frac{d^3}{dx^3}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x-3\sqrt{x}+3)}{8x^{\frac{5}{2}}},$$

$$\frac{d^4}{dx^4}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x^{\frac{3}{2}}-6x+15\sqrt{x}-15)}{16x^{\frac{7}{2}}}.\quad\therefore x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}=\frac{e^{\sqrt{x}}(x-3\sqrt{x}+3)}{8}.$$

$$\frac{d}{dx}\{x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}\}=\tfrac{1}{16}e^{\sqrt{x}}(\sqrt{x}-1),\ 2^5\frac{d^2}{dx^2}\{x^{\frac{5}{2}}\frac{d^3}{dx^3}e^{\sqrt{x}}\}=e^{\sqrt{x}}.$$

$$\text{Also } x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}=\tfrac{1}{16}e^{\sqrt{x}}(x^{\frac{3}{2}}-6x+15\sqrt{x}-15).$$

$$\frac{d}{dx}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=\tfrac{1}{32}e^{\sqrt{x}}(x-3\sqrt{x}+3),$$

$$\frac{d^2}{dx^2}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=\tfrac{1}{64}e^{\sqrt{x}}(\sqrt{x}-1),\ 2^7\frac{d^3}{dx^3}\{x^{\frac{7}{2}}\frac{d^4}{dx^4}e^{\sqrt{x}}\}=e^{\sqrt{x}}.$$

Hence generally $2^{2n+1}\dfrac{d^n}{dx^n}\{x^{n+\frac{1}{2}}\dfrac{d^{n+1}}{dx^{n+1}}e^{\sqrt{x}}\}=e^{\sqrt{x}}$.

Also solved by *B. F. YANNEY*.

PROBLEMS.

70. Proposed by J. A. CALDERHEAD, A. B., Professor of Mathematics, Curry University, Pittsburg, Pennsylvania.

Given $\sqrt[3]{a+x}+\sqrt[3]{a-x}=\sqrt[3]{c}$ to find x.

71. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

When $x=0$, find the the limit of the expression

$$U=\left(\frac{m+x}{m-x}\right)^{\frac{1}{x}}+\left(\frac{m-x}{m+x}\right)^{\frac{1}{x}}.$$

GEOMETRY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

54. Proposed by I. J. SCHWATT, Ph. D., University of Pennsylvania, Philadelphia, Pennsylvania.

Prove geometrically :

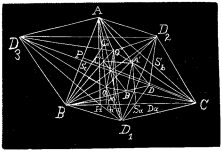
If through the center of perspective D of a given triangle ABC and its Brocard triangle $A'B'C'$ be drawn straight lines so as to pass through S_a, S_b and S_c (S_a, S_b, and S_c are the middle points of the sides BC, AC, and AB of the triangle ABC) and if S_aA_1 is made equal to DS_a, S_bD_2 equal to DS_b, and S_cD_3 equal to DS_c then are (1) the figures $D_1O'AO$, $D_2O'BO$ and $D_3O'CO$ parallelograms (O and O' are Brocard's points), (2) the triangles $D_1D_2D_3$ and ABC are equal, and (3) D_1A, D_2B, and D_3C intersect in S, (S is the middle point of OO').

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Since AC, DD_2 and BC, DD_1 bisect each other the quadrilaterals $ADCD_2$ and $BDCD_1$ are parallelograms, and AD_2, BD_1 both being equal and parallel to DC are equal and parallel to each other. Hence ABD_1D_2 is a parallelogram and AB is equal and parallel to D_1D_2. Similarly, AC is equal and parallel to D_1D_3, and BC is equal and parallel to D_2D_3.

$\therefore$ Triangle ABC is equal to triangle $D_1D_2D_3$. Also AD_1, BD_2, and

CD_3 intersect at the same point. For BD_2 and CD_3 bisect each other, also BD^2 and AD_1 bisect each other.



$\therefore BD_2$, AD_1, and CD_3 bisect one another in the same point. Since triangle BDC =triangle D_3AD_2, DD_a=the perpendiculer distance from A to D_2D_3.

Draw AH, OO_a, $O'O'_a$, DD_a perpendicular to BC; then the point of intersection of the three lines AD_1, BD_2, CD_3 is distant from BC, $\frac{1}{2}(AH-DD_a)$.

$$DD_a=\frac{2b^2c^2.\triangle}{a(a^2b^2+a^2c^2+b^2c^2)}.\quad \text{(Schwatt's Curves, p. 10).}$$

$$AH.a=2\triangle.\quad \therefore AH=\frac{2\triangle}{2a}.$$

$$\frac{AH-DD_a}{2}=\frac{\triangle.a(b^2+c^2)}{(a^2b^2+a^2c^2+b^2c^2)}=\frac{OO_a+O'O'_a}{2}.\quad \text{(Schwatt's Curves, p. 9).}$$

$\therefore AD_1$, BD_2, CD_3 intersect at the mid-point of OO'.

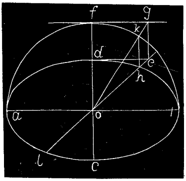
$\therefore$ Since AD_1, OO'; BD_2, OO'; CD_3, OO' all bisect one another, the quadrilaterals AOD_1O', BOD_2O', COD_3O' are parallelograms.

55. Proposed by FREDERICK R. HONEY, Ph. B., New Haven, Connecticut.

Let ab and cd be respectively the major and minor axes of an ellipse, and let α be the angle which a diameter lh forms with the major axis; it is required to find the length of this diameter.

I. Solution by the PROPOSER.

SOLUTION. Draw the semicircle afb on the diameter ab. Produce cd to f, and draw the tangents to the ellipse and the circle parallel to ab at the points d and f respectively. Produce lh to intersect de at e. Draw eg perpendicular to ab intersecting fg at g. Draw go intersecting the semicircle at k. Draw kh perpendicular to ab intersecting oe at h one extremity of the diameter lh.



ANALYSIS. The semiellipse adb may be considered as the projection on the plane of the paper of the semicircle afb, the latter being revolved about the diameter ab into a position when f is projected at d. The tangent fg which is parallel to ab is projected at de also parallel to ab. The points e and h are respectively the projections of g and k. Since the projection of every point on the

semicircle is found in a line drawn through it perpendicular to ab, the axis about which the semicircle revolves, kh drawn perpendicular to ab intersects oe at h and gives one point of the ellipse ; and therefore one extremity of the diameter lh.

II. Solution by H. C. WHITAKER, A. M., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania; W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, Louisiana; and J. O. MAHONEY, B. E., Assistant in Mathematics, Vanderbuilt University, Nashville, Tennessee.

If we denote ab by a, cd by b and tangent α by m, we have the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } y=mx \text{ which intersect at}$$

$$\left(\frac{ab}{\sqrt{b^2+a^2m^2}},\ \frac{abm}{\sqrt{b^2+a^2m^2}}\right) \text{ and } \left(-\frac{ab}{\sqrt{b^2+a^2m^2}},\ -\frac{abm}{\sqrt{b^2+a^2m^2}}\right),$$

the distance between these points being equal to

$$2ab\sqrt{\frac{1+m^2}{b^2+a^2m^2}}.$$

Also solved by *G. B. M. ZERR*, *J. SCHEFFER*, and *WILLIAM HOOVER*.

PROBLEMS.

60. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Onio University, Athens, Ohio.

Prove that the loci of the foci of the variable ellipses form a pair of circles passing through the extremities of the major axis of the fixed ellipse and having for diameters the semi-latasrectum of the fixed ellipse.

61. Proposed by WILLIAM E. HEAL, Member of the London Mathematical Society, and Treasurer of Grant County, Marion, Indiana.

Let the bisectors of the angles A, B, C of a triangle intersect in O and meet the sides opposite A, B, C in A', B', C'. Prove that the perpendiculars form O on the sides of the triangle $A'B'C'$ are $p_1=\frac{rR}{d_1}$, $p_2=\frac{rR}{d_2}$, $p_3=\frac{rR}{d_3}$ where r, R are the radii of the inscribed and circumscribed circles of the triangle ABC and d_1, d_2, d_3 are the distances of the center of the circumscribed circle from the centers of the three escribed circles.

62. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Prove that two triangles are equal if they have two sides and *the median of one of them* equal, each to each.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

49. Proposed by B. F. BURLESON, Oneida Castle, New York.

Find (1) in the leaf of the strophoid whose axis is a the axis of an inscribed leaf of the lemniscate, the node of the former coinciding with the crunode of the latter. Find (2) in a leaf of the lemniscate whose axis is b the axis a of an inscribed leaf of the strophoid, the node of the former also coinciding with the crunode of the latter.

Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Massachusetts.

Solving the equatians, $r\cos\theta+a\cos2\theta=0$ (strophoid), and $r=e^2\cos2\theta$ (lemniscate), we find they coincide when $\sin\theta=\sqrt{\tfrac{1}{2}}$ (1), or $\sin\theta=\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}$ (2).

(1) shows that they coincide at the origin for all values of a and e. We have to find the relation between the axes a and e which will make the curves tangent at the points determined by (2), provided those points are on both the leaves. Let $\phi=\angle$ made by the tangent at any point, with the radius vector drawn to that point. Then by the formula $\tan\phi=r\dfrac{d\theta}{dr}$.

Now for the lemniscate $r=\pm e\sqrt{\cos2\theta}$. $\dfrac{dr}{d\theta}=\dfrac{\mp e\sin2\theta}{\sqrt{\cos2\theta}}$.

$$\tan\phi=\pm e\sqrt{\cos2\theta}\left(\frac{\sqrt{\cos2\theta}}{\mp e\sin2\theta}\right)=\frac{2\sin^2\theta-1}{2\sin\theta\sqrt{1-\sin^2\theta}}\ \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

For the strophoid $r=-a\cos2\theta/\cos\theta$.

$$dr/d\theta=-a(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)/\cos^2\theta.$$

$$\tan\phi=[a\cos^2\theta\cos2\theta]/[a\cos\theta(-2\cos\theta\sin2\theta+\cos2\theta\sin\theta)]$$

$$=[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]\ \ldots\ldots\ldots\ldots(4).$$

Now equate (3) and (4) and substitute from (1),

$$[\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)]/[\sin\theta(2\sin^2\theta-3)]=(2\sin^2\theta-1)/2\sin\theta\sqrt{1-\sin^2\theta}.$$

$$2\sin\theta(1-\sin^2\theta)(1-2\sin^2\theta)=\sin\theta(1-2\sin^2\theta)(3-2\sin^2\theta),$$

$$2\sqrt{\tfrac{1}{2}}(1-\tfrac{1}{2})(1-1)=\sqrt{\tfrac{1}{2}}(1-1)(3-1)\text{ or }0=0,$$

which shows that the curves are tangent at point ($\theta=\sin^{-1}\sqrt{\tfrac{1}{2}}$, $r=0$) for any value of a and e. Again substituting from (2),

$$2\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{a^2}{2a^2-e^2}\right)\left(\frac{e^2}{2a^2-e^2}\right)=\sqrt{\frac{a^2-e^2}{2a^2-e^2}}\left(\frac{e^2}{2a^2-e^2}\right)\left(\frac{4a^2-e^2}{2a^2-e^2}\right).$$

This resolves into the three equations: $\sqrt{\dfrac{a^2-e^2}{2a^2-e^2}}=0$, whence $e=\pm a$..... (5);

$$\frac{e^2}{2a^2-e^2}=0,\text{ whence } e=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6);$$

$$\frac{2a^2}{2a^2-e^2}=\frac{4a^2-e^2}{2a^2-e^2},\text{ whence } e=\pm a\sqrt{2}\ldots\ldots\ldots\ldots\ldots (7).$$

From (5) substituted in (2), $\sin\theta=0$. $\therefore$ the curves are tangent at the extremity of the common axis, and the equations become,

$$r\cos\theta+a\cos2\theta=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$r^2=a^2\cos2\theta\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

From (9) $r_1=\pm a\sqrt{\cos2\theta}$.

From (8) $r_2=\dfrac{-a\cos2\theta}{\cos\theta}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{\cos2\theta}{1-\sin^2\theta}}=-a\sqrt{\cos2\theta}\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$.

Since for any value of $\sin\theta$ numerically less than $\sqrt{\tfrac{1}{2}}$, $\sqrt{\dfrac{1-2\sin^2\theta}{1-\sin^2\theta}}$ is numerically less than 1, r_2 is then numerically less than r_1. But by tracing the curves the leaf of each is seen to be formed by values of θ determined by this limit. $\therefore$ every point of the leaf of the strophoid lies within the lemniscate, and the former is in this case inscribed. From (6) equation of lemniscate becomes $r^2=0$, and the curve becomes a point. From (7) by substituting in (2) $\sin\theta=\sqrt{\dfrac{-a^2}{0}}$ an impossible value.

Accordingly the leaf of the strophoid can be inscribed in the leaf of the lemniscate when their axes are equal, and under no condition can the leaf of the lemniscate with an axis greater than 0 be inscribed in the leaf of the strophoid.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

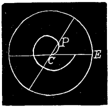
[It will be seen that Professor Black's result does not realize the intention of the problem as given by the Proposer. However, even for the Proposer's reading of the problem, his solution seems to us to be defective in several points. We may give Professor Zerr's solution later. EDITOR.]

50. Proposed by GEORGE LILLEY, Ph. D., LL. D., Principal of Park School, 394 Hall Street, Portland, Oregon.

A draw bridge, a feet in length, moves uniformly about a center axis. At the instant it began to open, a man stepped on the end; and, walking at a uniform rate in the straight line passing through its center, reached the opposite end just as it made n complete revolutions. Find the absolute path described by the man, and the ratio of his rate of motion in this path and the velocity of the end of the bridge. Apply the result to the case when a=320, and n=2.

Solution by E. L. SHERWOOD, A. M., Professor of Mathematics in Mississippi Normal College. Houston, Mississippi.

Let the man start at C and walk toward E, the table turning positively. He will traverse R, while the table turns $\frac{n}{2}.2\pi$. As velocities are uniform, we have,



$CP : PCE :: R : \pi n$, or $\rho : \theta :: R : \pi n$,

whence $\rho=\dfrac{R\theta}{\pi n}$ is the equation of the curve.

As $dl=[(\rho d\theta)^2+d\rho^2]^{\frac{1}{2}}$ we have,

$$dl=\frac{R}{\pi.n}(1+\theta^2)^{\frac{1}{2}}\,d\theta \text{ for this curve, and}$$

$$2l=\frac{2R}{\pi.n}\int_0^{\theta'}[1+\theta^2]^{\frac{1}{2}}\,d\theta, \text{ and}$$

$$L=\frac{2R}{\pi.n}\left[\frac{\theta}{2}(1+\theta^2)^{\frac{1}{2}}+\tfrac{1}{2}\log(\theta+\sqrt{1+\theta^2}\,)\right]_0^{\pi.n}$$

$$L=\frac{R}{\pi.n}[\pi.n(1+\pi^2.n^2)^{\frac{1}{2}}+\log(\pi.n+\sqrt{1+\pi^2.n^2})].$$

If a=100 feet, and n=2,

$$L=\frac{50}{2\pi}[2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})]=338.3.$$

[No. 41, *Calculus.*]

If a=36 inches, and n=1, we have,

$$L=\frac{18}{\pi}[\pi\sqrt{1+\pi^2}+\log(\pi+\sqrt{1+\pi^2}]=69.6+ \text{ inches.}$$

[No. 45, *Calculus.*]

If a=320, and n=2, we have,

$$L=\frac{160}{2\pi}[2\pi\sqrt{1+4\pi^2}+\log(2\pi+\sqrt{1+4\pi^2})]=1082.56 \text{ feet.}$$

[No. 50, *Calculus.*]

The ratio of rates of extremity of the bridge and the man in his path is:

$$\frac{a}{2}\,d\theta\div dl=\frac{\pi n}{\sqrt{1+\theta^2}}.$$

The ratio of rates of extremity of bridge and the man's *walking* is:

$$\frac{\pi an}{a}=\pi n.$$

Also solved by *G. B. M. ZERR* and *C. W. M. BLACK.*

PROBLEMS.

57. Proposed by F. M. McGAW. A. M., Mathematical Department, Bordentown Military Institute, Bordentown, New Jersey.

Solve the following equation : $(1+x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0.$

58. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A line passes through a fixed point and rotates uniformly about this point. Another line passes through a point which moves uniformly along the arc of a given curve and rotates uniformly about this point. Develop a method for finding the locus of intersection of these two lines. Apply to case of circle and straight line.

MECHANICS.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

32. Proposed by OTTO CLAYTON, A. B., Fowler, Indiana.

The wheel of a wind pump has 60 fans, each turned at an angle of 45° to the direction of the axis, and each having 150 square inches exposed to the wind. If the wind blows with a velocity of V and the wheel rotates with velocity ω, what is the component of force or pressure along the axis if it is turned at an angle α to the direction of the wind, assuming the resistance of the wheel in turning to be R?

No solution of this problem has been received.

33. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

At what angle with the axis of a stalk must a *sharp* wedge-shaped blade be struck, in order to *sever* the stalk with the *least* force?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let φ be the inclination of the axis of the stalk to the blade. A=area of of section made by blade, r=radius of stalk, and suppose resistance per unit of area to vary as $f(\varphi)$.

$\therefore R$ = resistance per unit of area $=mf(\varphi)$.

$\therefore A=\pi r^2 \operatorname{cosec}\varphi$.

$\therefore$ Work of cutting any section is $\pi r^2 mf(\varphi)\operatorname{cosec}\varphi$. This may be made a minimum when $f(\varphi)$ is known.

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

Since the blade is *sharp* we may neglect the force required to cut through the fibres and only regard that required to produce longitudinal compression.

Call k the coefficient of longitudinal compression, θ the angle of the blade, and ϕ the angle which the lower surface of the blade makes with the horizontal. Then when the blade has just cut through the stalk the force upon each surface parallel to the axis of the stalk will be

$$dF=k[\tan(\phi+\theta)-\tan\phi]xydx.$$

Resolving these parallel to the surface of wedge and parallel to median line of wedge we have—

$$dF_1=k.\frac{\cos(\phi+\theta)+\cos\phi}{\sin\frac{\theta}{2}}(\tan(\phi+\theta)-\tan\phi)xydx,$$

where F_1 is the force perpendicular to base of wedge.

$$\text{Then } F_1=k.\frac{\cos(\phi+\theta)+\cos\phi)}{\sin\frac{\theta}{2}}[\tan(\phi+\theta)-\tan\phi]\int_o^{2a}xydx$$

$$=2k\cos\frac{\theta}{2}\left[\sec(\phi+\theta)+\sec\phi\right]\int_o^{2a}xydx.$$

$$\frac{dF_1}{d\phi}=0 \text{ for minimum.}$$

$\therefore \sec(\phi+\theta)\tan(\phi+\theta)+\sec\phi\tan\phi=0$. By some obvious reductions

$$\sin(\phi+\frac{\theta}{2}).\{\cos^2(\phi+\frac{\theta}{2})+\sec^2\frac{\theta}{2}\}=0,$$

whence $\phi=-\frac{\theta}{2}$.

That is, the medial line is horizontal. The second factor gives imaginary results, except when $\theta=0$.

PROBLEMS.

39. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

A person whose height is a and weight W stands in a swing whose length is l. Supposing the initial inclination of the swing to the vertical is α and that the person always crouches when in the highest position and stands up when in the lowest, his center of gravity moving through a distance b measured from lower part of swing upward, find how much the arc is increased after n complete vibrations.

40. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the law of the force, in order that the orbit may be a Cassinian Oval.

41. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

If the earth were a perfect sphere and had a frictionless surface, what would be the motion of a ball placed at a given latitude?

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

The sum of three positive integral *cubic* roots of an equation is a square. What is the equation?

I. Solution by E. L. SHERWOOD, A. M., Professor of Mathematics and Science in Mississippi Normal College, Houston, Mississippi.

Let a, b, and c be the roots of the equation.

We then have $a^3+b^3+c^3=\square$.

This condition is satisfied by the equation $v^4(v^2+8v^2+27v^2)=\square$, where

$a^3=v^6$, $b^3=8v^6$ and $c^3=27v^6$. Forming the equation from the roots, we have: $x^3-(a^3+b^3+c^3)x^2+(a^3b^3+a^3c^3+b^3c^3)x-a^3b^3c^3=0$.

Substituting values of a, b, c and reducing, we have:

$x^3-36v^6x^2+251v^{12}x-216v^{18}=0$, where "$v$" may be 1, 2, 3, etc., in succession.

II. Solution by A. H. HOLMES, Brunswick, Maine, and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics in Texarkana College, Texarkana, Arkansas-Texas.

Let a, b, c be the roots of the cubic equation.

$\therefore x^3-(a+b+c)x^2+(ab+ac+bc)x=abc$, is the equation.

Let $a=5m^2$, $b=3m^2$, $c=m^2$. $\therefore 5m^2+3m^2+m=9m^2$.

$\therefore x^3-9m^2x^2+23m^4x=15m^6 \ldots\ldots\ldots(1)$.

Let $a=m^2+mn$, $b=n^2-mn$, $c=2mn$, $m>n$.

$\therefore m^2+mn+n^2-mn+2mn=(m+n)^2$.

$\therefore x^3-(m+n)^2x^2+(3m^3n+3mn^3)x=2m^4n^2-2m^2n^4\ldots\ldots\ldots\ldots(2)$.

(1) and (2) both satisfy the conditions.

41. Proposed by H. C. WILKES, Skull Run, West Virginia.

Given $\dfrac{50(a+b)}{ab}=\dfrac{81(c+d)}{cd}\ldots\ldots(1)$; $\dfrac{56(a+c)}{ac}=\dfrac{75(b+d)}{bd}\ldots\ldots(2)$;

$\dfrac{65(b+c)}{bc}=\dfrac{66(a+d)}{ad}\ldots\ldots(3)$, to find the least integral values of a, b, c, d.

I. Solution by the PROPOSER.

The sum of equations (1), (2) and (3), after clearing of fractions, can be reduced to $20d(ab+ac+bc)=111abc\ldots\ldots(4)$.

Eliminating from (1) and (4), $6d=9c$.

Eliminating from (2) and (4), $5d=9b$.

Eliminating from (3) and (4), $4d=9a$.

$\therefore$ The numbers are in the ratio $a4$, $b5$, $c6$, $d9$, which will be the least integers that will satisfy the equation. [See problem No. 36.]

II. Solution by A. H. BELL, Hillsboro, Illinois.

Arranging,	$50acd+50bcd=81abc+81abd$.	(1).
	$75acd+56bcd=-75abc+56abd$.	(2).
	$65acd-66bcd=66abc-65abd$.	(3).
$(1)\times 3$	$150acd+150bcd=243abc+243abd$.	(4).
$(2)\times 2$	$150acd-112bcd=-150abc+112abd$.	(5).
$(4)-(5)$	$262bcd=393abc+131abd$.	(6).
$(2)\times 13$	$975acd-728bcd=-975abc+728abd$.	(7).
$(3)\times 15$	$975acd-990bcd=990abc-975abd$.	(8).
$(7)-(8)$	$262bcd=-1965abc+1703abd$.	(9).

$(9)-(6)$, and reducing $3c=2d$. $\therefore c=2$, and $d=3$ (10).

These values in (1) and (2), etc., $a=4$ and $b=5$ (11).

To obtain the relative values between the two sets of values (10) and (11), take $(6)\times1703-(9)\times131$, results in $9a=4d$. $\therefore$ $a=4$ and $d=9$, $b=5$ and $c=6$. These are prime to each other. $\therefore$ are the least values.

III. Solution by COOPER D. SCHMITT, M. A., Professor of Mathematics, University of Tennessee, Nashville, Tennessee.

The equations can be written: $50\left(\frac{1}{a}+\frac{1}{b}\right)=81\left(\frac{1}{c}+\frac{1}{d}\right)$,

$$56\left(\frac{1}{c}+\frac{1}{a}\right)=75\left(\frac{1}{b}+\frac{1}{d}\right),\quad 65\left(\frac{1}{b}+\frac{1}{c}\right)=66\left(\frac{1}{a}+\frac{1}{d}\right).$$

Let $1/a=x$, $1/b=y$, $1/c=z$, and $1/d=u$, and the equations become $50x+50y-81z-81u=0$; $56x-75y+56z-75u=0$; $66x-65y-65z+66u=0$.

Thus we have three equations with four unknown quantities.

By determinants $x:y:z:u::$

$$\begin{vmatrix} 50, & -81, & -81 \\ -75, & 56, & -75 \\ -65, & -65, & 66 \end{vmatrix} : - \begin{vmatrix} 50, & -81, & -81 \\ 56, & 56, & -75 \\ 66, & -65, & 66 \end{vmatrix} : \begin{vmatrix} 50, & 50, & -81 \\ 56, & -75, & -75 \\ 66, & -65, & 66 \end{vmatrix} : - \begin{vmatrix} 50, & 50, & -81 \\ 56, & -75, & 56 \\ 66, & -65, & -65 \end{vmatrix}$$

Evaluating the determinants, we have,

$x:y:z:u::(131)^2 90:(131)^2 72:(131)^2 60:(131)^2 40$,

or $x:y:z:u::90:72:60:40$.

Hence $1/a:1/b:1/c:1/d::90:72:60:40$,

or $a:b::c:d::4:5:6:9$;

whence $a=4$, $b=5$, $c=6$, $d=9$ are the lowest values.

Also solved by *A. H. HOLMES*.

PROBLEMS.

47. Proposed by EDMUND FISH, Hillsboro Illinois.

A rectangular field, whose length and breadth in rods are in whole numders, is enclosed with a fence and subdivided by fences on both diagonals, the total length of the fences is 2204 rods; required the sides and area.

48. Proposed by SYLVESTER ROBBINS, North Branch Depot, New Jersey.

The edges of a rectangular parallelopiped are within 1 of the proportion $2:3:9$, and they are $2x\pm1$, $3x$ and $9x$, $(2x\mp1)^2+(3x)^2+(9x)^2$=the diagonal squared$=94x^2\mp4x+1=\square$. To find four integral values for x.

AVERAGE AND PROBABILITY.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

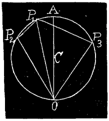
SOLUTIONS OF PROBLEMS.

30. Proposed by F. P. MATZ, M. A., M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

Find the average area of all the triangles which can be inscribed in a given circle.

I. Solution by the PROPOSER.

Let P_1OP_2 be any inscribed triangle; and through O draw any diameter OA. Two cases have now to be considered: (1), the triangle may lie *wholly* on one side of the diameter OA; (2), the triangle may lie *partly* on one side of the diameter OA.



I. Put $OA=2r$, $\angle AOP_1=\phi$, and $\angle AOP_2=\theta$; then $OP_1=2r\cos\phi$, $OP_2=2r\cos\theta$, and the area of the $\triangle$, P_1OP_2, $=A'$, $=2r^2\cos\phi\cos\theta\sin(\phi-\theta)$. Hence the average area of the triangles in this case, is

$$A_1=\int_0^{\frac{1}{2}\pi}\int_0^{\phi}A'd\phi d\theta\div\int_0^{\frac{1}{2}\pi}\int_0^{\phi}d\phi d\theta=\frac{8r^2}{\pi^2}\int_0^{\frac{1}{2}\pi}\phi\sin\phi\cos\phi d\phi$$

$$=\frac{r^2}{\pi^2}\Big[\sin2\phi-2\phi\cos2\phi\Big]_0^{\frac{1}{2}\pi}=\frac{r^2}{\pi}\ldots\ldots\ldots\ldots(1).$$

II. Put $\angle AOP_3=\psi$; then the area of the triangle P_2OP_3, $=A''$, $=2r^2\cos\theta\cos\psi\sin(\theta+\psi)$. Hence the average area of the triangles in this case, is

$$A_2=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}A''d\theta d\psi\div\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}d\theta d\psi=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta d\theta$$

$$+\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta\Big]=\frac{8r^2}{\pi^2}\Big[\frac{\pi}{4}\times\tfrac{1}{2}+\tfrac{1}{2}\times\frac{\pi}{4}\Big]=\frac{2r^2}{\pi}\ldots\ldots\ldots\ldots(2).$$

Hence the *required* average area becomes

$$A=\tfrac{1}{2}(A_1+A_2)=3r^2/2\pi\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots(3).$$

II. Solution by O. W. ANTHONY, M. Sc., Professor of Mathematics and Astronomy, New Windsor College, New Windsor, Maryland.

We readily get the area of triangle

$$=\frac{R^2}{2}\{\sin2A+\sin2B+\sin2C\},$$

which, by virtue of the relation $A+B+C=\pi$, reduces to

$$\frac{R^2}{2}\{\sin 2A+\sin 2B-\sin 2(A+B)\}.$$

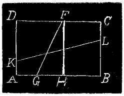
$$\therefore\ \text{Average area}=\frac{R^2}{2}\frac{\int_0^\pi\int_0^{\pi-A}\{\sin 2A+\sin 2B-\sin 2(A+B)\}dA\,dB}{\int_0^\pi\int_0^{\pi-A}dA\,dB}=\frac{3R^2}{2\pi}.$$

31. Proposed by B. F. FINKEL, A. M., Professor of Mathematics and Physics in Drury College, Springfield, Missouri.

Find the average length of a line drawn across the opposite sides of a rectangle, length l and breadth b.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas; and the PROPOSER.

Let $ABCD$ be the rectangle, FG the random line. Let $AB=l$, $BC=b$, $AH=x$, $AG=y$.



Then $FG=\{b^2+(x-y)^2\}^{\frac{1}{2}}$.

The limits of x are 0 and l; of y, 0 and x.

Hence the required average area is

$$\varDelta=\frac{\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}\,dxdy}{\int_0^l\int_0^x dxdy}$$

$$=\frac{2}{l^2}\int_0^l\int_0^x\{b^2+(x-y)^2\}^{\frac{1}{2}}\,dxdy$$

$$=\frac{1}{l^2}\int_0^l\{x(b^2+x^2)^{\frac{1}{2}}+b^2\log[x+(b^2+x^2)^{\frac{1}{2}}]-b^2\log b\}dx$$

$$=\frac{1}{3l^2}(l^2+b^2)^{\frac{3}{2}}+\frac{b^2}{l}\log\{l+(l^2+b^2)^{\frac{1}{2}}\}-\frac{b^2}{l}\log b-\frac{1}{l^2}(l^2+b^2)^{\frac{1}{2}}-\frac{b^3}{3l^2}+\frac{b}{l^2}.$$

For the line KL, we get, by writing l for b and b for l,

$$\varDelta_1=\frac{1}{3b^2}(l^2+b^2)^{\frac{3}{2}}+\frac{l^2}{b}\log\{b+(l^2+b^2)^{\frac{1}{2}}\}-\frac{l^2}{b}\log l-\frac{1}{b^2}(l^2+b^2)^{\frac{1}{2}}-\frac{l^3}{3b^2}+\frac{l}{b^2}.$$

Cor. I. If $l=b$, $\varDelta=\frac{1}{3}(2l\sqrt{2})+l\log(1+\sqrt{2})-\frac{1}{l}\sqrt{2}-\frac{1}{3}l+\frac{1}{l}$.

Cor. II. If $l=b=1$, $\varDelta=\frac{1}{3}(2-\sqrt{2})+\log(1+\sqrt{2})$, which is the same result as given in *Williamson's Integral Calculus*, page 409.

Also solved by *F. P. MATZ.*

PROBLEMS.

39. Proposed by O. W. ANTHONY, M. Sc., Professor of Mathematics in New Windsor College, New Windsor, Maryland.

A man is at the center of a circular desert; he travels at a given rate but in a *perfectly* random manner. What is the probability that he will be off the desert in a given time?

40. Proposed by HENRY HEATON, M. Sc., Atlantic, Iowa.

If every point of an ellipse be joined with every other point, what is the average length of the chords thus drawn?

41. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pennsylvania.

A line is drawn at random across the chord and *given* arc of a circular segment. Find the mean area of the *divisions*.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

35. Proposed by WILLIAM SYMMONDS, A. M., Professor of Mathematics and Astronomy in Pacific College, Santa Rosa, California; P. O., Sebastopol, California.

To an observer whose latitude is 40 degrees north, what is the sidereal time when Fomalhout and Antares have the same altitude: taking the Right Ascension and Declination of the former to be 22 hours, 52 minutes, —30 degrees, 12 minutes; of the latter 16 hours, 23 minutes, —26 degrees, 12 minutes?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let λ=latitude of observer, α, δ, α_1, δ_1 the Right Ascension and Declination of Fomalhout and Antares, respectively, β=altitude, h, h_1 the hour angles.

$\therefore \sin\beta=\sin\lambda\sin\delta+\cos\lambda\cos\delta\cos h=\sin\lambda\sin\delta_1+\cos\lambda\cos\delta_1\cos h_1$.

Also $h+\alpha=h_1+\alpha_1$.

But $\lambda=40^\circ$, $\alpha=343^\circ$, $\alpha_1=245^\circ\ 45'$, $\delta=-30^\circ\ 12'$, $\delta_1=-26^\circ\ 12'$.

$\therefore 66207\cos h-68734\cos h_1=3954$............(1).

$h_1-h=\alpha-\alpha_1=97^\circ\ 15'$, $\cos(h_1-h)=.12620$............(2).

Let $\cos h=x$, $\cos h_1=y$.

$\therefore$ from (2) $y=.1262x\pm\sqrt{.98407-.98407x^2}$. This in (1) gives $57532.7692x \mp 68184.81534\sqrt{1-x^2}=3954$.

$\therefore x^2-.05716x=.58216$, $\therefore x=.79211$ or $-.73495$.

$\therefore h=37^\circ 37'$ or $137^\circ 18'12''$. $h=2$ hours, 30 minutes, 28 seconds, or 9 hours, 9 minutes, 12.8 seconds.

$\therefore$ sidereal time $=1$ hour, 22 minutes, 28 seconds, or 8 hours, 1 minute, 12.8 seconds.

36. Proposed by J. K. ELLWOOD, A. M., Principal of the Colfax School, Pittsburg, Pennsylvania.

"What is the length of a chord cutting off one-fifth of the area of a circle whose diameter is 10 feet?"

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Applied Science, Texarkana College, Texarkana, Arkansas-Texas.

Let the chord subtend an angle $=2\theta$, $a=$radius of circle. Then the length of the chord $=2a\sin\theta$.

$\therefore a^2(\theta-\sin\theta\cos\theta)=\frac{1}{5}\pi a^2$.

$\therefore \theta-\sin\theta\cos\theta=\frac{1}{5}\pi$, $\therefore \theta=60^\circ 32'$ nearly.

$\therefore$ chord $=2a\sin\theta=10\sin\theta=8.7064$ feet.

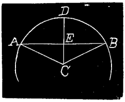
II. Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio, and Prof. P. S. BERG, Larimore, North Dakota.

Let $\theta=$the angle at the center, subtended by the required chord. Then $10\sin\theta=$the length of the required chord. Now $\frac{2\theta}{360}\pi 25$, the area of the sector, $-5\sin\theta\sqrt{(25-25\sin^2\theta)}$, the area of the triangle, $=5\pi$, the given area of the segment. Whence, by reduction, $\frac{\theta}{180}\pi-\sin\theta\cos\theta=\frac{\pi}{\cdot 5}$.

$\therefore \frac{\theta}{90}\pi-2\sin\theta\cos\theta=\frac{2}{5}\pi$. $\therefore .0349065\theta-\sin2\theta=1.256637$.

From which we readily find, by supposition, the value of θ; and from this, the value of $10\sin\theta$ to be 8.706, the length of the chord required.

III. Solution by A. H. BELL, Hillsboro, Illinois.



By Reversion of Series. Let the given diameter $=10=D$ and $1/5$ of circle $=a\pi r^2/d$, radius $=r$. To obtain the greatest convergency in the series, let ACB, the angle at the center $=2\theta$ and take the sector $ACD=r^2\theta/2$ and $r^2\sin\theta\cos\theta/2=ACE$.

Then $r^2(\theta-\sin\theta\cos\theta)/2=a\pi r^2/2d$ or $\text{arc}\theta=a\pi/d+\cos\theta\sqrt{(1-\cos^2\theta)}$ ....(1).

Make $\cos\theta=x$, and when expanded,

$$\theta=\frac{a\pi}{d}+x-\frac{x^3}{2}-\frac{x^5}{2.4}-\frac{3x^7}{2.4.6}-\frac{3.5x^9}{2.4.6.8}, \text{ etc.}, \ldots\ldots\ldots\ldots(2).$$

By trigonometry or calculus, we have,

$$\text{arc}\theta=\frac{\pi}{2}-x-\frac{x^3}{2.3}-\frac{3x^5}{2.4.5}-\frac{3.5x^7}{2.4.6.7}-\frac{3.5.7x^9}{2.4.6.8.9}, \text{ etc.}, \ldots\ldots(3).$$

(2)−(3) and ÷ by 2, etc.,

$$y=\frac{(d-2a)\pi}{4d}=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-\text{ etc.,}\ldots\ldots\ldots(4).$$

Assume $x=Ay+By^3+Cy^5+Dy^7+Ey^9+$ etc.,......................(5).

The powers of x substituted in (4), $y=Ay+$

$$\left(B-\frac{A^5}{6}\right)y^3+\left(C-\frac{A^2B}{2}-\frac{A^5}{40}\right)y^5+\left(D-\frac{A^2C}{2}-\frac{AB^2}{2}-\frac{A^4B}{8}-\frac{A^7}{112}\right)y^7+\text{etc.}$$

$\therefore$ $A=1$, $B=1/6$, $C=13/120$, $D=493/5040$, $E=37369/362880$, etc., in (5).
$x=\cos\theta=y+y^3/6+13y^5/120+493y^7/5040+37369y^9/362880+$ etc,,....(A).

Substituting values, $y=3\pi/20=0.471239=$logarithm $\bar{1}.673241+$.

2nd$=0.017441$
3rd$=0.002517$
4th$=0.000505$
5th$=0.000118$
Estimated$=0.000025$
$\cos\theta=0.491845$

2nd term $y^3=\bar{1}.019724-$	3rd term $y^5=\bar{2}.366206$
6 0.778151	$13/120=\bar{1}.034762$
$0.017441=\bar{2}.241573$	$0.002517+=\bar{3}.400968$
4th term $y^7=\bar{3}.712688$	5th term $y^9=\bar{3}.059171$
$493/5040$....$\bar{2}.990416$	$37369/362880$....$\bar{1}.012737$
$0.000505-=\bar{4}.703104$	$0.00018=\bar{4}.071908$

Chord $AB=10\sqrt{(1-\cos^2\theta)}=8.7068+$. $ACD=60°32'17''$ nearly.

NOTE.—Formula (A) is also a general solution for the height of the circular segment (see problem 37, page 75, Vol. II). When the angle ACD is less than 50°, solve (1) for $\sin\theta$, and we have,

$$\theta<50°=\sin\theta=\left(\frac{3a\pi}{2d}\right)^{\frac{1}{3}}-\tfrac{1}{10}\left(\frac{3a\pi}{2d}\right)^{\frac{3}{3}}-\tfrac{19}{1400}\left(\frac{3a\pi}{2d}\right)^{\frac{5}{3}}-\tfrac{71}{25200}\left(\frac{3a\pi}{2d}\right)^{\frac{7}{3}}-\text{, etc.,..}(B).$$

Chord$=D.\sin\theta$. It will be noticed that the convergency, in part, depends on the smallness of the value of y.

PROBLEMS.

42. Proposed by E. B. ESCOTT, 6123 Ellis Avenue, Chicago, Illinois.

To find a triangle whose sides and median lines are commensurable.

43. Proposed by H. C. WILKES, Skull Run, West Virginia.

To find, if possible, a right angled triangle, the bisectors of the acute angles of which, can be expressed by integral whole numbers.

44. Proposed by Prof. P. S. BERG, Larimore, North Dakota.

Two trees whose heights are 40 and 80 feet, respectively, stand on opposite sides of a stream 30 feet wide. What path does a squirrel take in leaping from the top of the higher to the top of the lower? What is the length of the path?

EDITORIALS.

The August-September number of the MONTHLY will be issued about the 20th of September.

The address of Editor Finkel, after July 1st, will be The University of Chicago, Chicago, Illinois.

This issue has been delayed on account of our engravers missending the plate for Mr. Miller's portrait.

The University of Pennsylvania has conferred the degree of Doctor of Philosophy on our valued contributor, H. C. Whitaker. We congratulate Prof. Whitaker on this merited recognition of his ability.

PERIODICALS.

Annual Recreation Number of the Outlook. The Outlook Publishing Co., 13 Astor Place, New York City.

The Outlook's seventh annual Recreation Number contains nearly a hundred pages and scores of illustrations. Nearly all of the special articles relate to outdoor life, sport, recreation, and vacation possibilities. Among the writers are Ian Maclaren, the Rev. Dr. Henry van Dyke, the Rev. Dr. Charles H. Parkhurst, Kirk Munroe, General A. W. Greely, Poultney Bigelow, and many others.

The Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. Review of Reviews Co., New York City.

The June number of *The Review of Reviews* is, as usual, full of the history of the important events that are taking place in various parts of the world. Dr. Shaw, the editor, has given a close analysis of the political situation which is now being worked out at St. Louis.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson, New York.

The June number of *The Cosmopolitan* is keeping up its literary merit, but is each time improving in the artistic excellence which it embodies.